

# Uncertainty Relation on Wigner-Yanase-Dyson Skew Information

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**Abstract.** We give a trace inequality related to the uncertainty relation of Wigner-Yanase-Dyson skew information. This inequality corresponds to a generalization of the uncertainty relation derived by S.Luo [6] for the quantum uncertainty quantity excluding the classical mixture.

**Key Words:** Uncertainty relation, Wigner-Yanase-Dyson skew information

## 1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_{\rho}(H) &= \frac{1}{2} \text{Tr} \left[ (i [\rho^{1/2}, H])^2 \right] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^{1/2} H \rho^{1/2} H] \end{aligned}$$

was defined in [8]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state  $\rho$  and an observable  $H$ . Here we denote the commutator by  $[X, Y] = XY - YX$ . This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho, \alpha}(H) &= \frac{1}{2} \text{Tr}[(i[\rho^{\alpha}, H])(i[\rho^{1-\alpha}, H])] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^{\alpha} H \rho^{1-\alpha} H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of  $I_{\rho, \alpha}(H)$  with respect to  $\rho$  was successfully proven by E.H.Lieb in [5]. From the physical point of view, an observable  $H$  is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we

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consider  $H \in B(\mathcal{H})$  represents the set of all bounded linear operators on the Hilbert space  $\mathcal{H}$ , as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by  $\mathcal{L}_h(\mathcal{H})$  and the set of all density operators (quantum states) by  $\mathcal{S}(\mathcal{H})$  on the Hilbert space  $\mathcal{H}$ . The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [7]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 9]. In our paper [9], we defined a generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we discuss various properties of the Wigner-Yanase-Dyson skew information. Finally in section3, we give our main result and its proof.

## 2 Trace inequalities of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable  $H$  in a quantum state  $\rho$  is expressed by  $Tr[\rho H]$ . It is natural that the variance for a quantum state  $\rho$  and an observable  $H$  is defined by  $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$ . It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \quad (2.1)$$

for a quantum state  $\rho$  and two observables  $A$  and  $B$ . The further strong results was given by Schrodinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A, B)|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by  $Cov_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$ . However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [7, 4, 9])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity  $U_\rho(H)$  representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (2.2)$$

then he derived the uncertainty relation on  $U_\rho(H)$  in [6]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2. \quad (2.3)$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (2.4)$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In this section, we study one-parameter extended inequality for the inequality (2.3).

**Definition 2.1** For  $0 \leq \alpha \leq 1$ , a quantum state  $\rho$  and an observable  $H$ , we define the Wigner-Yanase-Dyson skew information

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \end{aligned} \quad (2.5)$$

and we also define

$$\begin{aligned} J_{\rho,\alpha}(H) &= \frac{1}{2} \text{Tr}[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \end{aligned} \quad (2.6)$$

where  $H_0 = H - \text{Tr}[\rho H]I$  and we denote the anti-commutator by  $\{X, Y\} = XY + YX$ .

Note that we have

$$\frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2} \text{Tr}[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2} \text{Tr}[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \quad (2.7)$$

since we have  $\text{Tr}[\rho^{1/2} H \rho^{1/2} H] \leq \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H]$ . (See [1, 2] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \quad (2.8)$$

as a direct generalization of Eq.(2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H) \quad (2.9)$$

due to the first inequality of (2.7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4),(2.8),(2.9), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).$$

Our concern is to show an uncertainty relation with respect to  $U_{\rho,\alpha}(H)$  as a direct generalization of the inequality (2.3) such that

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \quad (2.10)$$

On the other hand, we introduced a generalized Wigner-Yanase skew information which is a generalization of the inequality (2.10), but different from the Wigner-Yanase-Dyson skew information defined in (2.5) and gave the following theorem in [10].

**Theorem 2.1** *For  $0 \leq \alpha \leq 1$ , a quantum state  $\rho$  and an observable  $H$ , we define a generalized Wigner-Yanase skew information by*

$$K_{\rho,\alpha}(H) = \frac{1}{2}Tr \left[ \left( i \left[ \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0 \right] \right)^2 \right]$$

and we also define

$$L_{\rho,\alpha}(H) = \frac{1}{2}Tr \left[ \left( i \left\{ \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0 \right\} \right)^2 \right],$$

and

$$W_{\rho,\alpha}(H) = \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)}.$$

Then for a quantum state  $\rho$  and observables  $A, B$  and  $\alpha \in [0, 1]$ , we have

$$W_{\rho,\alpha}(A)W_{\rho,\alpha}(B) \geq \frac{1}{4} \left| Tr \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [A, B] \right] \right|^2.$$

### 3 Main Theorem

We give the main theorem as follows;

**Theorem 3.1** *For a quantum state  $\rho$  and observables  $A, B$  and  $0 \leq \alpha \leq 1$ , we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1 - \alpha)|Tr[\rho[A, B]]|^2. \quad (3.1)$$

We use the several lemmas to prove the theorem 3.1. By spectral decomposition, there exists an orthonormal basis  $\{\phi_1, \phi_2, \dots\}$  consisting of eigenvectors of  $\rho$ . Let  $\lambda_1, \lambda_2, \dots$  be the corresponding eigenvalues, where  $\sum_{i=1}^{\infty} \lambda_i = 1$  and  $\lambda_i \geq 0$ . Thus,  $\rho$  has a spectral representation

$$\rho = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i|. \quad (3.2)$$

**Lemma 3.1**

$$I_{\rho, \alpha}(H) = \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$

**Proof of Lemma 3.1.** By (3.2),

$$\rho H_0^2 = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i | H_0^2.$$

Then

$$Tr[\rho H_0^2] = \sum_{i=1}^{\infty} \lambda_i \langle \phi_i | H_0^2 | \phi_i \rangle = \sum_{i=1}^{\infty} \lambda_i \|H_0 | \phi_i \rangle\|^2. \quad (3.3)$$

Since

$$\rho^\alpha H_0 = \sum_{i=1}^{\infty} \lambda_i^\alpha |\phi_i\rangle \langle \phi_i | H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{1-\alpha} |\phi_i\rangle \langle \phi_i | H_0,$$

we have

$$\rho^\alpha H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\phi_i\rangle \langle \phi_i | H_0 | \phi_j \rangle \langle \phi_j | H_0.$$

Thus

$$\begin{aligned} Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] &= \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} \langle \phi_i | H_0 | \phi_j \rangle \langle \phi_j | H_0 | \phi_i \rangle \\ &= \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2. \end{aligned} \quad (3.4)$$

From (2.5), (3.3), (3.4),

$$\begin{aligned}
I_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \|H_0|\phi_i\rangle\|^2 - \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&= \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle\phi_i|H_0|\phi_j\rangle|^2.
\end{aligned}$$

□

**Lemma 3.2**

$$J_{\rho,\alpha}(H) \geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle\phi_i|H_0|\phi_j\rangle|^2.$$

**Proof of Lemma 3.2.** By (2.6), (3.3), (3.4), we have

$$\begin{aligned}
J_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \|H_0|\phi_i\rangle\|^2 + \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&= \sum_{i,j=1}^{\infty} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&= 2 \sum_{i=1}^{\infty} \lambda_i |\langle\phi_i|H_0|\phi_i\rangle|^2 + \sum_{i \neq j} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&= 2 \sum_{i=1}^{\infty} \lambda_i |\langle\phi_i|H_0|\phi_i\rangle|^2 + \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle\phi_i|H_0|\phi_j\rangle|^2 \\
&\geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle\phi_i|H_0|\phi_j\rangle|^2.
\end{aligned}$$

□

**Lemma 3.3** *For any  $t > 0$  and  $0 \leq \alpha \leq 1$ , the following inequality holds;*

$$(1 - 2\alpha)^2(t - 1)^2 - (t^\alpha - t^{1-\alpha})^2 \geq 0. \quad (3.5)$$

**Proof of Lemma 3.3.** If  $\alpha = 0$  or  $\frac{1}{2}$  or  $1$ , then it is clear that (3.5) is satisfied. Now we put

$$F(t) = (1 - 2\alpha)^2(t - 1)^2 - (t^\alpha - t^{1-\alpha})^2.$$

We have

$$F'(t) = 2(1 - 2\alpha)^2t - 2\alpha t^{2\alpha-1} - 2(1 - \alpha)t^{1-2\alpha} + 8\alpha(1 - \alpha).$$

And we also have

$$F''(t) = 2(1 - 2\alpha)^2 - 2\alpha(2\alpha - 1)t^{2\alpha-2} - 2(1 - \alpha)(1 - 2\alpha)t^{-2\alpha}$$

and

$$\begin{aligned} F'''(t) &= 4\alpha(1 - 2\alpha)(1 - \alpha)t^{-2\alpha-1} - 4\alpha(1 - 2\alpha)(1 - \alpha)t^{2\alpha-3} \\ &= 4\alpha(1 - 2\alpha)(1 - \alpha) \left( \frac{1}{t^{1+2\alpha}} - \frac{1}{t^{3-2\alpha}} \right). \end{aligned}$$

If  $\frac{1}{2} < \alpha < 1$ , then  $1 + 2\alpha > 3 - 2\alpha$ . Then it is easy to show that  $F'''(t) < 0$  for  $t < 1$  and  $F'''(t) > 0$  for  $t > 1$ . On the other hand if  $0 < \alpha < \frac{1}{2}$ , then  $1 + 2\alpha < 3 - 2\alpha$ . Then it is easy to show that  $F'''(t) < 0$  for  $t < 1$  and  $F'''(t) > 0$  for  $t > 1$ . Since  $F''(1) = 0$ , we can get  $F''(t) > 0$ . Since  $F'(1) = 0$ , we also have  $F'(t) < 0$  for  $t < 1$  and  $F'(t) > 0$  for  $t > 1$ . Since  $F(1) = 0$ , we finally get  $F(t) \geq 0$  for all  $t > 0$ . Therefore we have (3.5).  $\square$

**Proof of Theorem 3.1.** We put  $t = \frac{\lambda_i}{\lambda_j}$  in (3.5). Then we have

$$(1 - 2\alpha)^2 \left( \frac{\lambda_i}{\lambda_j} - 1 \right)^2 - \left( \left( \frac{\lambda_i}{\lambda_j} \right)^\alpha - \left( \frac{\lambda_i}{\lambda_j} \right)^{1-\alpha} \right)^2 \geq 0.$$

And we get

$$(1 - 2\alpha)^2(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 0$$

and

$$(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 4\alpha(1 - \alpha)(\lambda_i - \lambda_j)^2$$

and

$$(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 4\alpha(1 - \alpha)(\lambda_i - \lambda_j)^2. \quad (3.6)$$

Since

$$Tr[\rho[A, B]] = Tr[\rho[A_0, B_0]]$$

$$\begin{aligned}
&= 2i \operatorname{Im} \operatorname{Tr}[\rho A_0 B_0] \\
&= 2i \operatorname{Im} \sum_{i < j} (\lambda_i - \lambda_j) \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle \\
&= 2i \sum_{i < j} (\lambda_i - \lambda_j) \operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle,
\end{aligned}$$

$$\begin{aligned}
|\operatorname{Tr}[\rho[A, B]]| &= 2 \left| \sum_{i < j} (\lambda_i - \lambda_j) \operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle \right| \\
&\leq 2 \sum_{i < j} |\lambda_i - \lambda_j| |\operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle|.
\end{aligned}$$

Then we have

$$|\operatorname{Tr}[\rho[A, B]]|^2 \leq 4 \left\{ \sum_{i < j} |\lambda_i - \lambda_j| |\operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2.$$

By (3.6) and Schwarz inequality,

$$\begin{aligned}
&\alpha(1 - \alpha) |\operatorname{Tr}[\rho[A, B]]|^2 \\
&\leq 4\alpha(1 - \alpha) \left\{ \sum_{i < j} |\lambda_i - \lambda_j| |\operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\
&= \left\{ \sum_{i < j} 2\sqrt{\alpha(1 - \alpha)} |\lambda_i - \lambda_j| |\operatorname{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\
&\leq \left\{ \sum_{i < j} 2\sqrt{\alpha(1 - \alpha)} |\lambda_i - \lambda_j| |\langle \phi_i | A_0 | \phi_j \rangle| |\langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\
&\leq \left\{ \sum_{i < j} \{(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2\}^{1/2} |\langle \phi_i | A_0 | \phi_j \rangle| |\langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\
&\leq \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | A_0 | \phi_j \rangle|^2 \\
&\quad \times \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | B_0 | \phi_j \rangle|^2.
\end{aligned}$$

Then we have

$$I_{\rho, \alpha}(A) J_{\rho, \alpha}(B) \geq \alpha(1 - \alpha) |\operatorname{Tr}[\rho[A, B]]|^2.$$

We also have

$$I_{\rho, \alpha}(B) J_{\rho, \alpha}(A) \geq \alpha(1 - \alpha) |\operatorname{Tr}[\rho[A, B]]|^2.$$

Hence we have the final result (3.1).  $\square$

**Remark 3.1** We remark that (2.3) is derived by putting  $\alpha = 1/2$  in (3.1). Then Theorem 3.1 is a generalization of the result of Luo [6].



**Remark 3.2** We remark that Conjecture 2.3 in [10] does not hold in genaral. The Conjecture is (2.10). A counterexample is given as follows. Let

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha = 1/3.$$

We have

$$I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) = I_{\rho,\alpha}(B)I_{\rho,\alpha}(A) = 0.22457296 \dots$$

and  $|Tr[\rho[A, B]]|^2 = 1$ . These imply

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.22457296 \dots < \frac{1}{4}|Tr[\rho[A, B]]|^2 = 0.25.$$

On the other hand we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) > \alpha(1 - \alpha)|Tr[\rho[A, B]]|^2 = 0.2222222 \dots$$

We also give a counterexample for Conjecture 2.10 in [10]. The inequality

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[(\frac{\rho^\alpha + \rho^{1-\alpha}}{2})^2[A, B]]|^2$$

is not correct in general, because  $LHS = 0.22457296 \dots$ ,  $RHS = 0.23828105995 \dots$ .

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